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Two novel classes of solvable many-body problems of goldfish type with constraints

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Abstract

Two novel classes of many-body models with nonlinear interactions 'of goldfish type' are introduced. They are *solvable* provided the initial data satisfy a single constraint (in one case; in the other, two constraints), i.e., for such initial data the solution of their initial-value problem can be achieved via *algebraic* operations, such as finding the eigenvalues of given matrices or equivalently the zeros of known polynomials. *Entirely isochronous* versions of some of these models are also exhibited, i.e., versions of these models whose nonsingular solutions are *all completely periodic* with the *same* period.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction and main results

Long time ago the possibility was noted of using the *nonlinear* mapping among the zeros and the coefficients of a polynomial in order to identify *solvable* (classes of) many-body problems, characterized by *nonlinear* equations of motions of Newtonian type (with one-body and two-body, generally velocity-dependent, forces) [1, 2]. (*Terminology:* here and hereafter we denote as *solvable* any problem whose solution can be achieved by algebraic operations, such as finding the zeros of polynomials). The starting point of these developments is a *linear* PDE such as

$$\begin{aligned}\psi_{tt} - [a_1 - (N-1)a_6 z]\psi_t + [a_2 + a_3 z - 2(N-1)a_{10}z^2]\psi_z \\ - (a_4 + a_5 z + a_6 z^2)\psi_{zt} + (a_7 + a_8 z + a_9 z^2 + a_{10} z^3)\psi_{zz} \\ - N[a_3 + (N-1)(a_9 - a_{10} z)]\psi = 0.\end{aligned} \quad (1)$$

(*Notation*: hereafter N is an arbitrary positive integer ($N \geqslant 2$), the symbols a_j and (see below) A_j, B_j denote (*a priori* arbitrary) *constants*, and subscripted variables denote partial differentiations of the dependent variable $\psi(z, t)$; note incidentally that this PDE coincides, up to trivial notational changes, with equation (2.3.3-1) of [2]). One then notes that this PDE admits solutions which are (for all time) a monic polynomial of degree N in z, and introduces the N zeros $z_n(t)$ and the N coefficients $c_m(t)$ of such a polynomial solution

$$\psi(z,t) = \prod_{n=1}^{N}[z - z_n(t)] = z^N + \sum_{m=1}^{N} c_m(t) z^{N-m}. \tag{2}$$

It is then easily seen that the fact that $\psi(z, t)$ satisfies the linear PDE (1) implies that the coefficients $c_m(t)$ evolve according to the following system of *linear* ODEs [2]:

$$\begin{aligned}\ddot{c}_m &- (N+1-m)a_4\dot{c}_{m-1} - [(N-m)a_5 + a_1]\dot{c}_m + ma_6\dot{c}_{m+1} \\ &+ (N+1-m)(N+2-m)a_7c_{m-2} + (N+1-m)[(N-m)a_8 + a_2]c_{m-1} \\ &- m[(2N-1-m)a_9 + a_3]c_m + m(m+1)a_{10}c_{m+1} = 0.\end{aligned} \tag{3}$$

Likewise, it can be shown (see [2], or section 2 below) that the zeros $z_n(t)$ evolve according to the following system of *nonlinear* PDEs:

$$\begin{aligned}\ddot{z}_n &= a_1\dot{z}_n + a_2 + a_3z_n - 2(N-1)a_{10}z_n^2 \\ &\quad + \sum_{m=1, m\neq n}^{N} \big\{(z_n - z_m)^{-1}\big[2\dot{z}_n\dot{z}_m + (\dot{z}_n + \dot{z}_m)(a_4 + a_5z_n) \\ &\quad + a_6(\dot{z}_nz_m + \dot{z}_mz_n)z_n + 2\big(a_7 + a_8z_n + a_9z_n^2 + a_{10}z_n^3\big)\big]\big\}.\end{aligned} \tag{4}$$

Notation and *terminology*: hereafter indices such as m, n range from 1 to N (unless otherwise indicated), superimposed dots denote differentiations with respect to the time t, and the N zeros $z_n(t)$ are interpreted as the coordinates of N 'point particles' evolving according to the 'Newtonian equations of motion' (4) featuring one-body and two-body velocity-dependent forces; and an analogous interpretation shall be given in all analogous cases below. Unless otherwise explicitly specified, we hereafter assume that all quantities under consideration (except the time t) are generally *complex* numbers, so that the motions of the particles characterized by the coordinates $z_n(t)$ take place in the complex z-plane; of course such motions can easily be reinterpreted as taking place in a 'more physical' *real* plane, but we shall not devote space to discuss this well-known aspect (see, for instance, chapter 4 of [2], entitled 'Solvable and or integrable many-body problems in the plane, obtained by complexification').

Clearly, since the parameters a_j are (arbitrary) *constants*, the solution of the system of *linear* ODEs (3) satisfied by the coefficients $c_m(t)$ can be achieved by purely *algebraic* operations; and once the N coefficients $c_m(t)$ of the polynomial $\psi(z, t)$ have been obtained, the computation of the N zeros $z_n(t)$ of this polynomial, see (2), is also a purely *algebraic* task. Hence the determination of the N coordinates $z_n(t)$ whose time evolution is characterized by the Newtonian equations of motion (4) can be achieved via purely *algebraic* operations: this N-body problem is *solvable*.

The ten-parameter class of *solvable* N-body models (4) is vast, and it includes several interesting cases [2]. Even the very simplest case, characterized by the vanishing of *all* the parameters a_j, is quite remarkable: in this case the Newtonian equations of motion read simply

$$\ddot{z}_n = \sum_{m=1, m\neq n}^{N} \frac{2\dot{z}_n\dot{z}_m}{z_n - z_m}, \tag{5a}$$

and the solution of the *initial-value* problem for this problem is given by the following simple proposition: *the N coordinates $z_n(t)$ are the N zeros of the following algebraic equation in z*:

$$\sum_{n=1}^{N} \frac{\dot{z}_n(0)}{z - z_n(0)} = \frac{1}{t}. \tag{5b}$$

Note that, by multiplying this equation by the product $\prod_{n=1}^{N}[z - z_n(0)]$, it becomes a polynomial equation, with time-dependent coefficients, of degree N in z, which indeed generally has N solutions, at least if one allows the coordinates z_n to be *complex* numbers, hence their time evolution to take place in the *complex* z-plane; indeed such an evolution can also be reinterpreted as the motion of N-point particles on a *real* (physical) plane, and its phenomenology is quite amusing, see section 2.4.2 of [2], entitled 'The simplest model: explicit solution (the game of musical chairs), Hamiltonian structure'. Because of these remarkable features, as well as the neat form of the Newtonian equations of motion (5*a*) (which also happen to be Hamiltonian), this N-body model was given the honorary title of 'goldfish' [3], and this attribute was then extended to several *solvable* models featuring on the right-hand side of their Newtonian equations of motion (acceleration equal force) terms such as that appearing on the right-hand side of (5*a*): note that these *solvable* models 'of goldfish type' include now the models belonging to the ten-parameter class (4) and several models outside of this class (for an updated review of such models see [4]).

In the present paper we exhibit two classes of *solvable* N-body models 'of goldfish type' which are, to the best of our knowledge, *new*, hence, in particular, *not* included in the class (4). These novel models are however *solvable* only provided the particle coordinates $z_n(t)$ satisfy *additional* conditions, which must, of course, be compatible with the time evolution, so that it is sufficient that they be satisfied, in the context of the *initial-value* problem, by the *initial data*. This limitation could be considered as a drawback of these new models, or instead as a feature that adds to their interest in the context of mathematical physics. The hunch that such models should exist was suggested to us by recent results [7–11] concerning certain *exceptional polynomial subspaces*, as tersely reviewed in appendix A.

The first class of these models is characterized by the following (six-parameter) Newtonian equations of motion:

$$\ddot{z}_n = A_1\dot{z}_n + A_2 + A_3 z_n - \frac{2A_4}{z_n} + \sum_{m=1, m\neq n}^{N} \left\{(z_n - z_m)^{-1}\left[2\dot{z}_n\dot{z}_m + A_5(\dot{z}_n + \dot{z}_m)z_n \right.\right.$$
$$\left.\left. + 2\left(A_4 - A_2 z_n + A_6 z_n^2\right)\right]\right\}. \tag{6}$$

Here the *novel* element—not included in the class (4)—is that associated with the constant A_4. In the following section we show that these equations of motion entail that the coefficients $c_m(t)$—related to the coordinates $z_n(t)$ via the mapping (2)—evolve according to the following system of *linear* ODEs:

$$\ddot{c}_m - [(N-m)A_5 + A_1]\dot{c}_m + (N-1-m)(N+2-m)A_4 c_{m-2}$$
$$- (N+1-m)(N-1-m)A_2 c_{m-1}$$
$$- m[(2N-1-m)A_6 + A_3]c_m = 0, \tag{7}$$

of course with $c_0 = 1$ and $c_m = 0$ for $m > N$ and $m < 0$ (consistently with (2), and also with this system of ODEs). Note that the *linear* character of this system of ODEs entails the *solvable* character of the Newtonian equations of motion (6). But these findings are only valid if the coordinates $z_n(t)$ satisfy the constraint

$$\sum_{n=1}^{N} \frac{1}{z_n(t)} = 0, \tag{8}$$

and correspondingly (see below) the coefficients $c_m(t)$ satisfy the constraint

$$c_{N-1}(t) = 0. \tag{9}$$

The treatment of section 2 implies that these constraints, (8) respectively (9), are indeed compatible with the evolution equations (6) respectively (7); hence they are automatically satisfied, in the context of the *initial-value* problem, provided the initial data for the *N*-body problem (6) satisfy the following two conditions:

$$\sum_{n=1}^{N} \frac{1}{z_n(0)} = 0, \qquad \sum_{n=1}^{N} \frac{\dot{z}_n(0)}{[z_n(0)]^2} = 0, \tag{10}$$

and correspondingly the initial data for the *linear* system of ODEs (7) satisfy the two conditions

$$c_{N-1}(0) = \dot{c}_{N-1}(0) = 0. \tag{11}$$

Before turning to the second class of new models, let us display an equivalent avatar of the Newtonian equations of motion (6):

$$\begin{aligned}\ddot{z}_n = {} & \left(A_1 + \frac{N-2}{2}A_5\right)\dot{z}_n - (N-2)A_2 + [A_3 + (N-2)A_6]z_n - \frac{2A_4}{z_n} \\ & + \frac{A_5}{2}\dot{Z} + A_6 Z + \sum_{m=1, m\neq n}^{N} \left\{(z_n - z_m)^{-1}\left[2\dot{z}_n\dot{z}_m + \frac{A_5}{2}(\dot{z}_n + \dot{z}_m)(z_n + z_m)\right.\right. \\ & \left.\left. + 2A_4 - A_2(z_n + z_m) + A_6\left(z_n^2 + z_m^2\right)\right]\right\},\end{aligned} \tag{12}$$

where

$$Z(t) = \sum_{n=1}^{N} z_n(t). \tag{13}$$

This second version, (12) with (13), has the merit to immediately yield (by summing over n from 1 to N—using (8) and the vanishing of the double sum on the right-hand side due to the antisymmetry of the summand under the exchange of the dummy indices m and n) the linear ODE

$$\ddot{Z} = [A_1 + (N-1)A_5]\dot{Z} - N(N-2)A_2 + [A_3 + 2(N-1)A_6]Z,$$

which is consistent, via the relation

$$c_1(t) = -Z(t) \tag{14}$$

implied by (2), with the equation satisfied by $c_1(t)$ that obtains by setting $m = 1$ in (7). On the other hand, the first version, (6), has the merit to feature only one-body and two-body forces.

The second class of new models is characterized by the following (seven-parameter) Newtonian equations of motion:

$$\begin{aligned}\ddot{z}_n = {} & B_1\dot{z}_n - (N-1)B_2 z_n - 2(N-1)B_3 z_n^2 + B_4\frac{\dot{z}_n}{z_n} \\ & + \sum_{m=1, m\neq n}^{N} \left\{(z_n - z_m)^{-1}\left[2\dot{z}_n\dot{z}_m + (\dot{z}_n + \dot{z}_m)(B_4 + B_5 z_n)\right.\right. \\ & \left.\left. + B_6(\dot{z}_n z_m + \dot{z}_m z_n)z_n + 2\left(B_7 z_n + B_2 z_n^2 + B_3 z_n^3\right)\right]\right\}.\end{aligned} \tag{15}$$

Clearly the *novel* element (in (15) relative to (4)) is now that associated with the constant B_4. In the following section we show that these equations of motion, (15), entail that the coefficients $c_m(t)$—related to the coordinates $z_n(t)$ via the mapping (2)—evolve as follows:

$$\ddot{c}_m - (N-m)B_4\dot{c}_{m-1} - [(N-m)B_5 + B_1]\dot{c}_m + mB_6\dot{c}_{m+1} + (N+1-m)(N-m)B_7c_{m-1} \\ - m(N-m)B_2c_m + m(m+1)B_3c_{m+1} = 0, \tag{16}$$

entailing the *solvable* character of the Newtonian equations of motion (6). But this conclusion is valid only if the coordinates $z_n(t)$ satisfy the constraint

$$\sum_{n=1}^{N} \frac{\dot{z}_n(t)}{z_n(t)} = 0, \tag{17}$$

and correspondingly (see below) the coefficients $c_m(t)$ satisfy the constraint

$$\dot{c}_N(t) = 0. \tag{18}$$

The treatment of section 2 implies that these constraints are indeed compatible with the evolution equations (15) and (16); hence they are automatically satisfied, in the context of the *initial-value* problem, provided the initial data for the N-body problem (15) satisfy the following single condition:

$$\sum_{n=1}^{N} \frac{\dot{z}_n(0)}{z_n(0)} = 0, \tag{19}$$

and correspondingly the initial data for the *linear* system of ODEs (16) satisfy the single condition

$$\dot{c}_N(0) = 0. \tag{20}$$

The paper is organized as follows. In section 2 the results reported above are proven. In sections 3 and 4 we discuss tersely the behaviour of these two *solvable* N-body problems, and in section 5 *entirely isochronous* versions of these novel models are identified: they are characterized by the fact that *all* their nonsingular solutions are *completely periodic* (namely, periodic in *all* their dependent variables $z_n(t)$) with the *same* basic period (or possibly with a—generally not too large [12]—*integer multiple* of it). Section 6 outlines tersely further developments, to be reported in future publications. The paper is completed by two appendices: in the first one certain relevant results concerning *exceptional polynomial subspaces* are tersely reviewed; in the second one certain computations are confined whose treatment in the body of the paper would disrupt the flow of the presentation.

2. Proofs

The procedure to arrive at the results reported above is by now textbook material [2, 4], hence our treatment here can be terse; although the new twist should be emphasized, implying that the *solvable* character of the new models applies only provided their time evolution is somewhat restricted—as already indicated in the preceding section. The starting point of our treatment is the *linear* PDE (but see the remark below)

$$\psi_{tt} - [a_1 - (N-1)za_6]\psi_t + [a_2 + a_3z - 2(N-1)a_{10}]\psi_z \\ - (a_4 + a_5z + a_6z^2)\psi_{zt} + (a_7 + a_8z + a_9z^2 + a_{10}z^3)\psi_{zz} \\ - N[a_3 + (N-1)(a_9 - a_{10}z)]\psi \\ = \frac{a_{11}}{z}\left[\psi_t - \frac{\dot{c}_N}{c_N}\right] + \frac{a_{12}}{z}\left[\psi_z - \frac{c_{N-1}}{c_N}\right]. \tag{21}$$

(*Notation:* the symbols a_j denote again (*a priori* arbitrary) constants, subscripted variables denote partial differentiations and the dependent variable $\psi(z,t)$ as well as the coefficients $c_N(t)$ and $c_{N-1}(t)$ are characterized by (2)).

Remark. The consistency of this evolution PDE, (21), with the fact that the dependent variable $\psi(z,t)$ is the monic polynomial (2) of degree N in z shall be clear from what follows. Note however that this evolution equation, (21), is a *linear PDE* iff the two constants a_{11} and a_{12} vanish (in which case it coincides with (1)); otherwise it is in fact a *nonlinear functional* equation, as implied by the formulae (clearly entailed by (2))

$$c_N(t) = \psi(0,t), \qquad c_{N-1}(t) = \psi_z(0,t). \tag{22}$$

Via (2), this evolution equation (21) entails[4] the following (systems of) evolution ODEs for the N coefficients $c_m(t)$ and for the N zeros $z_n(t)$:

$$\begin{aligned}\ddot{c}_m &- (N+1-m)a_4\dot{c}_{m-1} - [(N-m)a_5 + a_1]\dot{c}_m + ma_6\dot{c}_{m+1}\\ &\quad + (N+1-m)(N+2-m)a_7c_{m-2} + (N+1-m)[(N-m)a_8 + a_2]c_{m-1}\\ &\quad - m[(2N-1-m)a_9 + a_3]c_m + m(m+1)a_{10}c_{m+1}\\ &= a_{11}\left(\dot{c}_{m-1} - \frac{\dot{c}_N}{c_N}c_{m-1}\right) + a_{12}\left[(N+2-m)c_{m-2} - \frac{c_{N-1}}{c_N}c_{m-1}\right],\end{aligned} \tag{23}$$

$$\begin{aligned}\ddot{z}_n &= a_1\dot{z}_n + a_2 + a_3z_n - 2(N-1)a_{10}z_n^2 + a_{11}\frac{\dot{z}_n}{z_n} - \frac{a_{12}}{z_n}\\ &\quad + \sum_{m=1,m\neq n}^{N}\left\{(z_n - z_m)^{-1}\left[2\dot{z}_n\dot{z}_m + (\dot{z}_n + \dot{z}_m)(a_4 + a_5z_n)\right.\right.\\ &\quad \left.\left. + a_6(\dot{z}_nz_m + \dot{z}_mz_n)z_n + 2\left(a_7 + a_8z_n + a_9z_n^2 + a_{10}z_n^3\right)\right]\right\}.\end{aligned} \tag{24}$$

The observation that system (23) is consistent with $c_0 = 1$ and $c_m = 0$ for $m > N$ and $m < 0$ confirms the consistency with the evolution equation (21) of the assumption that $\psi(z,t)$ be a monic polynomial of degree N in z, see (2).

In order to guarantee that these Newtonian equations of motion, (24), be *solvable*, we must make sure that the corresponding evolution equations (23) be effectively *linear*. A straightforward way to achieve this goal is to set to zero the two parameters a_{11} and a_{12}; but this simply reproduces the previously known (class of) *solvable* N-body models (4). There are however two other possibilities, which do yield two *new* classes of integrable systems.

The first possibility is to set $a_{11} = 0$ (but $a_{12} \neq 0$) and then to require that the system of evolution equations (23) be consistent with constraint (9). It is indeed plain that system (23) becomes then *linear*:

$$\begin{aligned}\ddot{c}_m &- (N+1-m)a_4\dot{c}_{m-1} - [(N-m)a_5 + a_1]\dot{c}_m + ma_6\dot{c}_{m+1}\\ &\quad + (N+1-m)(N+2-m)a_7c_{m-2} + (N+1-m)[(N-m)a_8 + a_2]c_{m-1}\\ &\quad - m[(2N-1-m)a_9 + a_3]c_m + m(m+1)a_{10}c_{m+1}\\ &= a_{12}(N+2-m)c_{m-2},\end{aligned} \tag{25}$$

and it is moreover easily seen (by setting $m = N-1$) that this system is consistent with constraint (9), provided the parameters a_j satisfy the following restrictions:

$$a_4 = a_6 = 2a_7 - a_{12} = a_8 + a_2 = a_{10} = a_{11} = 0, \tag{26}$$

[4] The derivation of the first of these two systems of ODEs is a rather trivial exercise; the derivation of the second system is as well trivial, but readers who find this second task exceedingly painstaking are advised to use the formulae given in section 2.3.2 of [2], or (even more conveniently) in appendix A of [4].

entailing that system (23) becomes

$$\ddot{c}_m - [(N-m)a_5 + a_1]\dot{c}_m + (N-1-m)(N+2-m)a_7 c_{m-2} - (N+1-m) \\ \times (N-1-m)a_2 c_{m-1} - m[(2N-1-m)a_9 + a_3]c_m = 0, \tag{27}$$

and likewise system (24) becomes

$$\ddot{z}_n = a_1\dot{z}_n + a_2 + a_3 z_n - \frac{2a_7}{z_n} + \sum_{m=1, m\neq n}^{N} \{(z_n - z_m)^{-1}[2\dot{z}_n\dot{z}_m + (\dot{z}_n + \dot{z}_m)a_5 z_n \\ + 2(a_7 - a_2 z_n + a_9 z_n^2)]\}. \tag{28}$$

It is easily seen that these systems coincide with (7) and (6) after the following trivial relabelling of the parameters: $a_j = A_j$ for $j = 1, 2, 3, 5$; $a_7 = A_4$, $a_9 = A_6$. And it is moreover plain that constraint (9) corresponds to constraint (8), since relation (2) among the zeros $z_n(t)$ and the coefficients $c_m(t)$ of the polynomial ψ clearly entails[5]

$$c_N(t) = (-)^N \prod_{n=1}^{N} z_n(t), \qquad c_{N-1}(t) = (-)^{N-1} \sum_{m=1}^{N} \prod_{n=1, n\neq m}^{N} z_n(t), \tag{29a}$$

hence

$$c_{N-1}(t) = -c_N(t) \sum_{n=1}^{N} \frac{1}{z_n(t)}. \tag{29b}$$

The second possibility is to set $a_{12} = 0$ (but $a_{11} \neq 0$) and then to require that the system of evolution equations (23) be consistent with constraint (18). It is indeed plain that system (23) becomes then *linear*:

$$\ddot{c}_m - [(N+1-m)a_4 - a_{11}]\dot{c}_{m-1} - [(N-m)a_5 + a_1]\dot{c}_m + m a_6 \dot{c}_{m+1} \\ + (N+1-m)(N+2-m)a_7 c_{m-2} + (N+1-m)[(N-m)a_8 + a_2]c_{m-1} \\ - m[(2N-1-m)a_9 + a_3]c_m + m(m+1)a_{10}c_{m+1} = 0, \tag{30}$$

and it is moreover easily seen (by setting $m = N$) that this system is consistent with constraint (18), provided the parameters a_j satisfy the following restrictions:

$$a_4 - a_{11} = a_7 = a_2 = (N-1)a_9 + a_3 = a_{12} = 0, \tag{31}$$

entailing that system (23) becomes

$$\ddot{c}_m - (N-m)a_4\dot{c}_{m-1} - [(N-m)a_5 + a_1]\dot{c}_m + m a_6\dot{c}_{m+1} + (N+1-m)(N-m)a_8 c_{m-1} \\ - m(N-m)a_9 c_m + m(m+1)a_{10}c_{m+1} = 0, \tag{32}$$

and likewise system (24) becomes

$$\ddot{z}_n = a_1\dot{z}_n - (N-1)a_9 z_n - 2(N-1)a_{10}z_n^2 + a_4\frac{\dot{z}_n}{z_n} \\ + \sum_{m=1, m\neq n}^{N} \{(z_n - z_m)^{-1}[2\dot{z}_n\dot{z}_m + (\dot{z}_n + \dot{z}_m)(a_4 + a_5 z_n) \\ + a_6(\dot{z}_n z_m + \dot{z}_m z_n)z_n + 2(a_8 z_n + a_9 z_n^2 + a_{10}z_n^3)]\}. \tag{33}$$

[5] Actually to reach these conclusions one needs the additional condition $c_N(t) \neq 0$, namely (see (29a)) $z_n(t) \neq 0$; but the fact that such a condition must be imposed initially, and that it shall subsequently hold for all regular solutions of the Newtonian equations of motion (6), is implied by the presence of the term proportional to A_4 on the right-hand side of these equations of motion.

It is easily seen that these systems coincide with (16) and (15) after the following trivial relabelling of the parameters: $a_j = B_j$ for $j = 1, 4, 5, 6$; $a_8 = B_7, a_9 = B_2, a_{10} = B_3$. And it is moreover plain[6] that constraint (18) corresponds to constraint (17), see (29*a*).

3. Behaviour of the *N*-body models belonging to the first class

In this section we discuss—in somewhat more detail than done in the introductory section 1—the behaviour of the solutions of N-body models belonging to the first class, see (6).

The *general* solution of the *linear* system of ODEs (7) with (9) is given by the formula

$$c_m(t) = \sum_{\ell=1}^{N} \left[\gamma_\ell^{(+)} v_m^{(\ell)(+)} \exp\left(\lambda_\ell^{(+)} t\right) + \gamma_\ell^{(-)} v_m^{(\ell)(-)} \exp\left(\lambda_\ell^{(-)} t\right)\right]. \tag{34}$$

Here the N-vectors $\underline{v}^{(\ell)(\pm)}$ are the $2N$ eigenvectors corresponding to the $2N$ eigenvalues $\lambda_\ell^{(\pm)}$ of the (generalized) eigenvalue equation

$$(\lambda^2 \underline{\mathbf{1}} + \lambda \underline{M}^{(1)} + \underline{M}^{(2)})\underline{v} = 0, \tag{35}$$

where the $N \times N$ matrices $\underline{M}^{(1)}$ and $\underline{M}^{(2)}$ are defined componentwise as follows (see (7)):

$$M_{m,m}^{(1)} = -[(N-m)A_5 + A_1], \tag{36a}$$
$$M_{m,m}^{(2)} = -m[(2N-m-1)A_6 + A_3], \tag{36b}$$
$$M_{m,m-1}^{(2)} = -(N+1-m)(N-1-m)A_2, \tag{36c}$$
$$M_{m,m-2}^{(2)} = (N-1-m)(N+2-m)A_4, \tag{36d}$$

with all other elements vanishing. As for the $2N$ constants $\gamma_\ell^{(\pm)}$, they are *arbitrary* except for the two requirements

$$\sum_{\ell=1}^{N} \left[\gamma_\ell^{(+)} v_{N-1}^{(\ell)(+)} + \gamma_\ell^{(-)} v_{N-1}^{(\ell)(-)}\right] = 0, \tag{37a}$$
$$\sum_{\ell=1}^{N} \left[\gamma_\ell^{(+)} v_{N-1}^{(\ell)(+)} \lambda_\ell^{(+)} + \gamma_\ell^{(-)} v_{N-1}^{(\ell)(-)} \lambda_\ell^{(-)}\right] = 0, \tag{37b}$$

which clearly correspond to constraints (11) via (34). Of course, in the context of the *initial-value* problem, the $2N$ constants $\gamma_\ell^{(\pm)}$ are determined by the $2N$ initial data $c_m(0), \dot{c}_m(0)$ (including (11)) via the system of $2N$ *linear* equations

$$c_m(0) = \sum_{\ell=1}^{N} \left[\gamma_\ell^{(+)} v_m^{(\ell)(+)} + \gamma_\ell^{(-)} v_m^{(\ell)(-)}\right], \tag{38a}$$
$$\dot{c}_m(0) = \sum_{\ell=1}^{N} \left[\gamma_\ell^{(+)} v_m^{(\ell)(+)} \lambda_\ell^{(+)} + \gamma_\ell^{(-)} v_m^{(\ell)(-)} \lambda_\ell^{(-)}\right]. \tag{38b}$$

And in the context of the *initial-value* problem for the Newtonian N-body model (6) the initial values $c_m(0), \dot{c}_m(0)$ are determined by the initial values $z_n(0), \dot{z}_n(0)$ via the following relations implied by (2):

$$\prod_{n=1}^{N} [z - z_n(0)] = z^N + \sum_{m=1}^{N} c_m(0) z^{N-m}, \tag{39}$$

[6] Again with the additional condition $c_N(t) \neq 0$, which need not be highlighted since the presence of the term proportional to B_4 on the right-hand side of the Newtonian equations of motion (15) guarantees automatically its validity for all regular solutions of this system of ODEs.

$$-\sum_{n=1}^{N} \dot{z}_n(0) \prod_{m=1, m\neq n}^{N} [z - z_m(0)] = \sum_{m=1}^{N} \dot{c}_m(0) z^{N-m}. \tag{40}$$

Remark. Let us emphasize that, while in the context of the *initial-value* problem for the N-body problem (6) the constants $\gamma_\ell^{(\pm)}$ depend on the initial data (as just explained), the $2N$ eigenvalues $\lambda_\ell^{(\pm)}$—characterizing via (34) the time evolution of the coefficients $c_m(t)$ of the polynomial $\psi(z, t)$ whose N zeros $z_n(t)$ yield the coordinates of the N moving particles—do *not* depend on the initial data, but only on the parameters A_j that specify the N-body model (6) under consideration.

We now note that the $N \times N$ matrix $\underline{M}^{(1)}$ is *diagonal*, and the $N \times N$ matrix $\underline{M}^{(2)}$ is *triangular* ($M_{nm}^{(2)} = 0$ if $n < m$). Hence the eigenvalues $\lambda_m^{(\pm)}$ are just the roots of the N (decoupled) second-order equations

$$\lambda^2 - [(N-m)A_5 + A_1]\lambda - m[(2N-m-1)A_6 + A_3] = 0, \tag{41}$$

i.e.

$$\lambda_m^{(\pm)} = \tfrac{1}{2}\{(N-m)A_5 + A_1 \pm \Delta_m\}, \tag{42a}$$

$$\begin{aligned}\Delta_m^2 &= [(N-m)A_5 + A_1]^2 + 4m[(2N-m-1)A_6 + A_3] \\ &= (NA_5 + A_1)^2 + \left(A_5^2 - 4A_6\right)m^2 \\ &\quad + 2[2A_3 + 2(2N-1)A_6 - A_5(A_1 + NA_5)]m.\end{aligned} \tag{42b}$$

The behaviour of the solutions of the N-body problem (6) with (8) is given by the time evolution of the N zeros $z_n(t)$ of the polynomial $\psi(z, t)$, see (2), whose coefficients evolve exponentially in time as entailed by (34) with (3). The study of the time evolution of the particle coordinates $z_n(t)$ is therefore reduced to the study of the motion of the zeros of a polynomial whose coefficients depend exponentially on time. In the generic case—characterized by exponents not all of which are purely imaginary—the asymptotic behaviour of these zeros in the remote past and future—characterizing the qualitative behaviour of the N-body model under consideration—can therefore be easily evinced by the treatment provided in appendix G (entitled ‘Asymptotic behaviour of the zeros of a polynomial whose coefficients diverge exponentially’) of [2]. The subclass of N-body models (6) characterized by parameters A_j satisfying the following restrictions,

$$A_1 = \mathrm{i}\alpha, \qquad A_3 = \gamma, \qquad A_5 = \mathrm{i}\beta, \qquad A_6 = \eta, \tag{43a}$$

$$[(N-m)\beta + \alpha]^2 - 4m[(2N-m-1)\eta + \gamma] > 0, \qquad m = 1, \ldots, N, \tag{43b}$$

(where the four constants $\alpha, \beta, \gamma, \eta$ are *all real*)—conditions which are necessary and sufficient to guarantee that *all* the $2N$ eigenvalues $\lambda_m^{(\pm)}$ are *imaginary* numbers—is instead clearly characterized by the property that *all* motions are *confined*. The more special subcase in which *all* the $2N$ eigenvalues $\lambda_m^{(\pm)}$ are *integer multiples* of a single quantity $\mathrm{i}\omega$ with ω a *positive* constant, $\omega > 0$, so that *all* the coefficients $c_m(t)$ are *periodic* with the *same* period $T = 2\pi/\omega$, is discussed in section 5.

4. Behaviour of the N-body models belonging to the second class

In this section, we discuss—in somewhat more detail than done in the introductory section 1; but rather tersely, to avoid repetitions of developments already elaborated in the preceding section 3—the behaviour of the solutions of N-body models belonging to the second class, see (15).

The *general* solution of the *linear* system of ODEs (16) with (18) is given by a formula entirely analogous to (34), and the subsequent developments are also analogous to those reported in the preceding section, except that now the $N \times N$ matrices $\underline{M}^{(1)}$ and $\underline{M}^{(2)}$ are defined componentwise as follows (see (16)):

$$M^{(1)}_{m,m+1} = mB_6, \tag{44a}$$

$$M^{(1)}_{m,m} = -[(N-m)B_5 + B_1], \tag{44b}$$

$$M^{(1)}_{m,m-1} = -(N-m)B_4, \tag{44c}$$

$$M^{(2)}_{m,m+1} = m(m+1)B_3, \tag{44d}$$

$$M^{(2)}_{m,m} = -m(N-m)B_2, \tag{44e}$$

$$M^{(2)}_{m,m-1} = (N+1-m)(N-m)B_7, \tag{44f}$$

with all other elements vanishing. As for the $2N$ constants $\gamma_\ell^{(\pm)}$, they are again *arbitrary* except now for the single requirement

$$\sum_{\ell=1}^{N} \left[\gamma_\ell^{(+)} v_N^{(\ell)(+)} \lambda_\ell^{(+)} + \gamma_\ell^{(-)} v_N^{(\ell)(-)} \lambda_\ell^{(-)}\right] = 0, \tag{45}$$

which clearly corresponds to constraint (20) via (34).

Since the $N \times N$ matrices $\underline{M}^{(1)}$ and $\underline{M}^{(2)}$ are now neither diagonal nor triangular, the computations of the eigenvalues $\lambda_m^{(\pm)}$ cannot be done now in explicit form (in the general case, with the coefficients B_j appearing in (4) unrestricted). Therefore, a discussion of the actual behaviour of the solutions of this second *solvable* model cannot be done here in this general case to the same explicit extent as in the case of the first class of models, treated in the preceding section. There is however a subcase of this second model for which the treatment becomes closely analogous to that of the preceding section. Indeed clearly if

$$B_3 = B_6 = 0, \tag{46}$$

the two $N \times N$ matrices $\underline{M}^{(1)}$ and $\underline{M}^{(2)}$ become both *triangular*, hence the corresponding eigenvalue equation becomes again a second-order algebraic equation that can be easily solved[7]. Indeed it is easily seen that if the restriction (46) holds, this second-order equation determining the exponents $\lambda_m^{(\pm)}$ that characterize the behaviour of the *second* class of models *coincides* with (a subcase of) the second-order equation (41) determining the exponents $\lambda_m^{(\pm)}$ that characterize the behaviour of the *first* class of models, provided one sets

$$A_1 = B_1, \qquad A_5 = B_5, \qquad A_6 = B_2, \qquad A_3 + (N-1)A_6 = 0, \tag{47}$$

hence the discussion given above for the first class of models becomes applicable to the second class of models provided they are restricted by condition (46). And this includes of course also the more special, *entirely isochronous*, subcase, as discussed in section 5.

[7] The $N \times N$ matrices $\underline{M}^{(1)}$ and $\underline{M}^{(2)}$ become triangular also in the case $B_4 = B_7 = 0$, but whenever B_4 vanishes the novelty of the case treated in this paper disappears, so we do not pursue this case.

5. Isochronous versions of the solvable models

A current definition of *isochronous* systems (see, for instance, [4]) attributes this property to any dynamical system that possesses at least one *open* (hence *fully-dimensional*) region in its phase space within which *all* solutions are *completely periodic* (i.e., periodic in *all* dependent variables) with the *same* fixed period (of course, the period being independent of the initial data, as long as they stay within that *isochronicity* region). The class of *isochronous* systems is vast [4], and it includes a large zoo of systems interpretable as N-body problems characterized by *autonomous* equations of motion of Newtonian type. A class of such *isochronous* systems is characterized by the Newtonian equations of motion (see example 4.1.2–3 in [4])

$$\ddot{\underline{z}} = -\mathrm{i}\omega\dot{\underline{z}} + \sum_{k=1}^{K} \underline{F}^{(-k)}(\underline{z}, \dot{\underline{z}} + \mathrm{i}\omega\underline{z}). \tag{48a}$$

Here underlined variables indicate N-vectors, ω is a *nonvanishing real* constant, K is an arbitrary *positive* integer and the K (N-vector-valued) functions $\underline{F}^{(-k)}(\underline{z}, \underline{\tilde{z}})$ are required to be *analytic* (but not necessarily *holomorphic*) in all their $2N$ arguments and to satisfy the scaling property

$$\underline{F}^{(-k)}(c\underline{z}, \underline{\tilde{z}}) = c^{-k}\underline{F}^{(-k)}(\underline{z}, \underline{\tilde{z}}), \qquad k = 1, \ldots, K. \tag{48b}$$

Indeed it has been shown [4] that these dynamical systems, (48), are *isochronous*, possessing an open (hence *fully dimensional*) region in their phase space in which *all* their solutions are *completely periodic* with period

$$T = \frac{2\pi}{\omega}. \tag{49}$$

It is easily seen that system (24) belongs to this class, (48), with $K = 1$, provided

$$\begin{aligned} &a_1 = -(2N-1)\mathrm{i}\omega, && a_2 = [a_{11} - (N-1)a_4]\mathrm{i}\omega,\\ &a_3 = 2(N-1)\omega^2, && a_5 = 2\mathrm{i}\omega,\\ &a_6 = 0, && a_8 = \mathrm{i}\omega a_4,\\ &a_9 = -\omega^2, && a_{10} = 0, \end{aligned} \tag{50}$$

while the constants a_4, a_7, a_{11} and a_{12} remain *arbitrary*. It is clear that both systems, (6) respectively (15), fall within this class, provided the constants A_j respectively B_j featured by these two *solvable* models satisfy appropriate restrictions (whose explicit determination can be left as a simple exercise for the diligent reader).

But here we like to use the more restrictive definition of *entire isochronicity* [4], stating that a dynamical system is *entirely isochronous* if *all* its nonsingular solutions are *completely periodic* with period T, or possibly with a (not arbitrarily large; indeed, generally rather small [12]) *integer* multiple of T—implying that *all* nonsingular solutions of an *entirely isochronous* system are in fact *completely periodic* with the *same* period, which however need not be the *primitive* period for all of them; or, equivalently, that the property of *isochronicity* holds in the *entire* phase space (with the possible exceptions of a lower dimensional set of initial data yielding *singular* solutions).

One can then assert (see appendix B for a proof) that the N-body problem (6) is indeed *entirely isochronous* provided the six constants A_j it features satisfy the following four restrictions:

$$A_1 = (k_1 - Nk_2)\mathrm{i}\omega, \tag{51a}$$

$$A_3 = \frac{1}{2}(k_2 + k_3)\left[\left(N - \frac{1}{2}\right)(k_2 - k_3) - k_1\right]\omega^2, \tag{51b}$$

$$A_5 = k_2 \mathrm{i}\omega, \tag{51c}$$

$$A_6 = \frac{k_3^2 - k_2^2}{4}\omega^2, \tag{51d}$$

where ω is a *positive* constant, $\omega > 0$ and the three numbers k_1, k_2, k_3 are *integers*, unrestricted (i.e., positive, negative or vanishing) except for the requirement

$$k_1 \neq -mk_3 \qquad \text{for} \quad m = 1, \ldots, N. \tag{51e}$$

Note that the two constants A_2 and A_4 remain completely arbitrary. Also note that, if the two integers k_1 and k_2 vanish—entailing that A_1 and A_5 vanish, $A_1 = A_5 = 0$, and $A_3 = (1-2N)A_6$, the only remaining restriction on A_6 being that it be a *positive* real number, $A_6 > 0$—then the equations of motion (6) become *real* (of course provided *real* values are also assigned to the two *a priori arbitrary* constants A_2 and A_4).

But there is a second, different class of *entirely isochronous* systems, that is obtained from (a subclass of) the *solvable* system (6) with (8) via the standard 'trick' procedure (see for instance [4]), as we now explain. To this end we take as a starting point the following special case of the model (6):

$$\zeta_n'' = -\frac{2A_4}{\zeta_n} + 2\sum_{m=1, m\neq n}^{N} \frac{\zeta_n'\zeta_m' + A_4}{\zeta_n - \zeta_m}, \tag{52}$$

which corresponds to (6) with *all* the constants A_j set to zero except A_4 (and moreover the purely notational replacement—convenient for what shall immediately follow—of the dependent variables $z_n(t)$ with the dependent variables $\zeta_n(\tau)$, and of the independent variable t with the independent variable τ, appended primes denoting of course differentiations with respect to this new independent variable τ). We then set

$$z_n(t) = \exp(-\mathrm{i}\omega t)\zeta_n(\tau), \qquad \tau = \frac{\exp(\mathrm{i}\omega t) - 1}{\mathrm{i}\omega}, \tag{53}$$

with ω being a *positive* constant, and we thereby obtain the new N-body model characterized by the (*autonomous*) Newtonian equations of motion

$$\ddot{z}_n = -\mathrm{i}\omega\dot{z}_n - \frac{2A_4}{z_n} + 2\sum_{m=1, m\neq n}^{N} \{(z_n - z_m)^{-1}[(\dot{z}_n + \mathrm{i}\omega z_n)(\dot{z}_m + \mathrm{i}\omega z_m) + A_4]\}. \tag{54}$$

Due to the way this model has been obtained it is clear that it describes an *entirely isochronous* N-body problem—provided its initial data are constrained by conditions (10), guaranteeing the validity of (8) (both for this model and for the model (52)). It is also easy to verify that this model is *not* a special case of the *entirely isochronous* N-body problem identified above, i.e. (6) with (51) (and of course with (8)).

Let us turn now to a discussion of *entirely isochronous* variants of the second class of solvable models treated in this paper. The identification of such models cannot be explicitly achieved in the general case along the same lines as done above—in the initial part of this section—for models belonging to the first class of solvable systems, because in this second case one cannot obtain explicitly the exponents $\lambda_m^{(\pm)}$, as explained in the preceding section (see the paragraph after (35)), unless the restriction (46) holds, in which case the treatment given in the first part of this section for the *first* class of models can be immediately extended to the *second* class via relations (47), which however entail (see the last of these equations (47)) that the restrictions (51) must now be complemented by the additional restriction

$$k_2 + k_3 = 0 \qquad \text{or} \qquad 2k_1 = N(k_2 - k_3). \tag{55}$$

It is moreover again possible, also for the models of the second class, to proceed via the standard trick procedure. Since this approach is quite analogous to that described immediately above for the *first* class, we deal with it quite tersely.

The starting point is now the *solvable* system (15) with all constants vanishing except B_4, which we now write as follows:

$$\zeta_n'' = B_4 \frac{\zeta_n'}{\zeta_n} + \sum_{m=1, m\neq n}^{N} \frac{2\zeta_n'\zeta_m' + B_4(\zeta_n' + \zeta_m')}{\zeta_n - \zeta_m}. \tag{56}$$

It is now again appropriate to use the version (53) of the trick, obtaining thereby the N-body problem—clearly *entirely isochronous*, because of the way it is obtained—characterized by the following equations of motion:

$$\ddot{z}_n = -\mathrm{i}\omega\dot{z}_n + B_4 \frac{\dot{z}_n + \mathrm{i}\omega z_n}{z_n} + \sum_{m=1, m\neq n}^{N} \frac{2(\dot{z}_n + \mathrm{i}\omega z_n)(\dot{z}_m + \mathrm{i}\omega z_m) + B_4[\dot{z}_n + \dot{z}_m + \mathrm{i}\omega(z_n + z_m)]}{z_n - z_m}. \tag{57}$$

6. Numerical results

In this section, we perform a numerical integration of the equations of motion (6) of the first model in order to illustrate the previous findings. The numerical integration has been done with an embedded Runge–Kutta method of order 8 (5, 3) with automatic step size control, as developed by Prince and Dormand [13]. The integration and the graphical output have been performed with the software Dynamics solver[8] developed by J Aguirregabiria.

In order to simplify our numerical treatment we have set in (6) $N = 3$, and $A_1 = A_2 = A_5 = 0$. Hence, our numerical study focuses on the following system of three nonlinear complex ODEs of second order:

$$\ddot{z}_n = A_3 z_n - \frac{2A_4}{z_n} + \sum_{m=1, m\neq n}^{3} (z_n - z_m)^{-1}\left[2\dot{z}_n\dot{z}_m + 2\left(A_4 + A_6 z_n^2\right)\right]. \tag{58}$$

Generic (chaotic) behaviour. When $A_4 = 0$, this system is solvable and it is a particular case of the family discussed in chapter 2.3.3 of [2]. When $A_4 \neq 0$ and no restrictions are imposed on the initial data, system (58) corresponds to the motion of the zeros of a third degree polynomial whose coefficients evolve in time according to the nonlinear system (23). The system behaves in general in a chaotic manner as can be evinced from the numerical results in figures 1(*e*), (*f*) and 2. The initial data for that integration have been chosen to be

$$\begin{aligned} z_1(0) &= 1.5 - 2\mathrm{i}, & \dot{z}_1(0) &= 2 - \mathrm{i} \\ z_2(0) &= -1 + 0\mathrm{i}, & \dot{z}_2(0) &= -1 + \mathrm{i} \\ z_3(0) &= 1.2 + 0.4\mathrm{i}, & \dot{z}_3(0) &= 2.32 - 0.76\mathrm{i}, \end{aligned} \tag{59}$$

which do not satisfy constraints (10), while the values of the coefficients are

$$A_3 = -5.1\pi^2, \qquad A_4 = 15, \qquad A_6 = \pi^2. \tag{60}$$

[8] This program is available at http://tp.lc.ehu.es/jma/ds/ds.html.

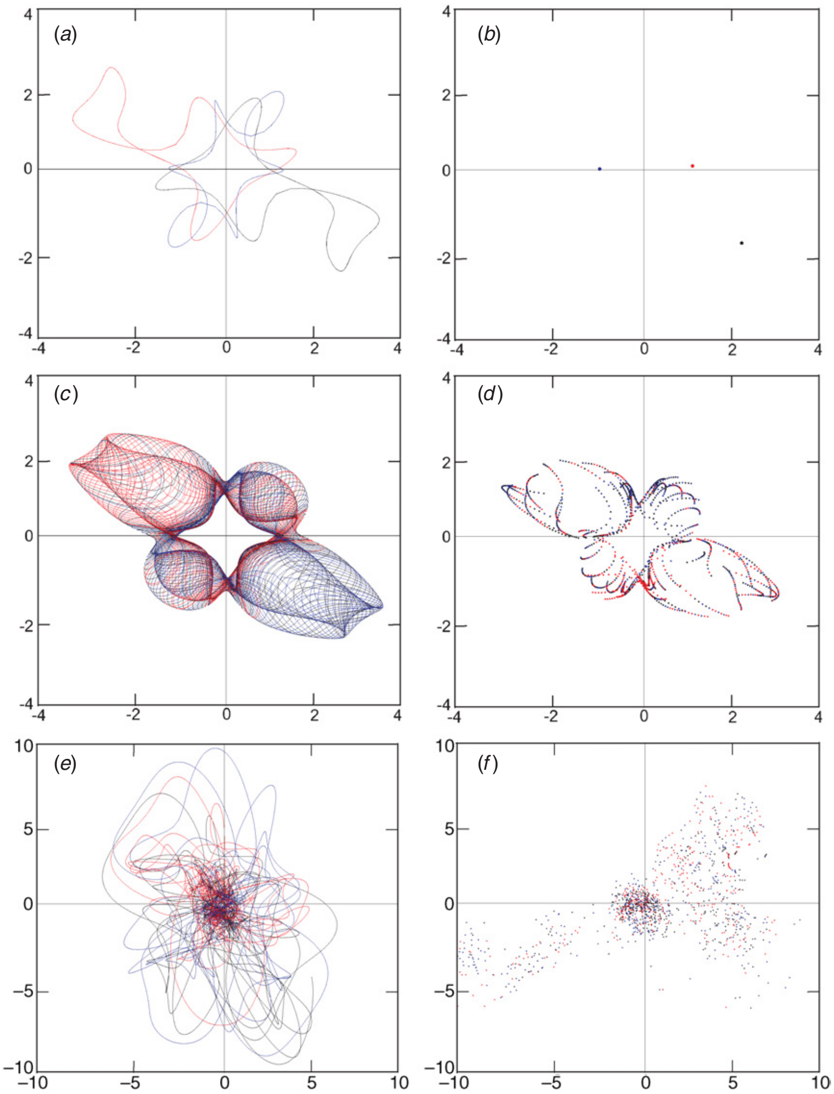

Figure 1. Left column: trajectories of the three-body system (58) in the periodic (a), quasi-periodic (c) or chaotic regimes (e). Right column: stereographic projection of the trajectories: the positions of the three particles are plotted at every integer multiple of $T = 1$ up to 10^3 iterations.

6.1. Quasi-periodic behaviour

However, for the same values of these parameters, (60), initial data can be chosen within the algebraic submanifold defined by (10). One such assignment is

$$\begin{aligned} z_1(0) &= 2 - 2\mathrm{i}, & \dot{z}_1(0) &= 2 - \mathrm{i}, \\ z_2(0) &= -1 + 0\mathrm{i}, & \dot{z}_2(0) &= -1 + \mathrm{i}, \\ z_3(0) &= 1.2 + 0.4\mathrm{i}, & \dot{z}_3(0) &= 2.32 - 0.76\mathrm{i}. \end{aligned} \tag{61}$$

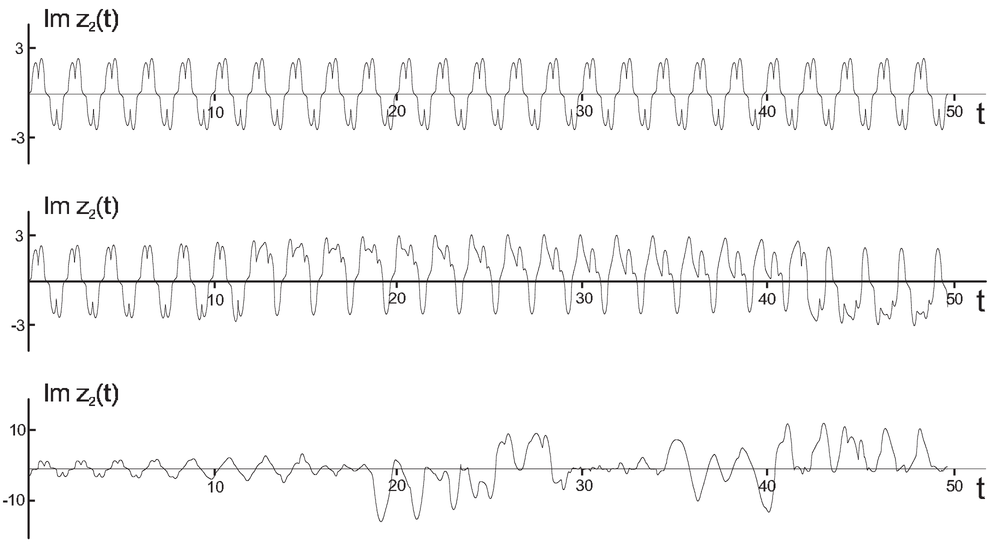

Figure 2. A plot of $\mathrm{Im}\, z_2(t)$ for the choice of parameters and initial data described in section 6. The periodic, quasi-periodic and chaotic behaviour are manifest.

As discussed in section 2, now system (58) is *solvable*, and it corresponds to the motion of the zeros of a monic third degree polynomial whose coefficients evolve in time according to the following linear system:

$$\ddot{c}_1 - (4A_6 + A_3)c_1 = 0, \tag{62a}$$

$$c_2 = 0, \tag{62b}$$

$$\ddot{c}_3 - 2A_4c_1 - 3(2A_6 + A_3)c_3 = 0. \tag{62c}$$

The solution of this system is trivial and from (3) and the discussion in section 4, we see that it suffices that

$$4A_6 + A_3 < 0, \qquad 2A_6 + A_3 < 0$$

for the orbits to be confined. As it follows from (34) with (3), the coefficients of the polynomial are in general quasi-periodic functions of time, hence also the behaviour of system (10) is quasi-periodic as can be seen in figures 1(*c*), (*d*) and 2, where we have again assigned the values (60) of the parameters, and the same initial data (61).

6.2. Isochronous behaviour

Following the discussion of section 5 and appendix B, the family of *solvable* problems discussed above includes a subclass of *entirely isochronous* systems. They occur when all the frequencies of the system are commensurable. A look at (51) reveals that these conditions are met if we choose

$$A_1 = A_5 = 0, \qquad A_3 = -\tfrac{5}{4}k_3^2\omega^2, \qquad A_6 = \tfrac{1}{4}k_3^2\omega^2. \tag{63}$$

We set $k_3 = 1$ and $\omega = 2\pi$ so that the fundamental period is $T = 1$. The choice of parameters is then

$$A_3 = -5\pi^2, \qquad A_4 = 15, \qquad A_6 = \pi^2, \tag{64}$$

and the initial data are the same as in the previous case (61). The resulting motions are shown in figures 1(*a*), (*b*) and 2.

7. Outlook

The novel cases treated in this paper do not exhaust all the possible *solvable* many-body problems associated with the motion of the roots of polynomials whose coefficients satisfy a linear system of ODEs. This question is related to the classification of exceptional subspaces X_k with co-dimension $k > 1$, which has not yet been completed. However, as mentioned in appendix A, some results based on X_2 spaces and operators will be reported soon [5].

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Appendix A

In this appendix, we explain the general method to write down certain PDEs in 1+1 dimensions possessing solutions which are monic polynomials $\psi(z,t)$ of degree n in z whose N coefficients are time-dependent functions $c_m(t)$. The requirement to ensure the solvability of the *nonlinear* N-body problem characterizing the time evolution of the N zeros $z_n(t)$ of such a polynomial is that the coefficients $c_m(t)$ evolve according to a *linear* system of ODEs.

Let us denote by $\mathcal{P}_N$ the vector space of polynomials in z of degree less than or equal to N:

$$\mathcal{P}_N = \operatorname{span}\{1, z, z^2, \ldots, z^N\}, \qquad \dim \mathcal{P}_N = N + 1. \tag{A.1}$$

We shall also denote by $\mathcal{D}_2(\mathcal{P}_N)$ the vector space of differential operators of order 2 in z with analytic coefficients that leave the space $\mathcal{P}_N$ invariant, i.e.

$$\mathcal{D}_2(\mathcal{P}_N) = \{T = a_2(z)D_{zz} + a_1(z)D_z + a_0(z) | T\mathcal{P}_N \subset \mathcal{P}_N\}. \tag{A.2}$$

Notation: here and below we use the notation $D_z \equiv \partial/\partial z$, $D_{zz} \equiv \partial^2/\partial z^2$, $D_t \equiv \partial/\partial t$, $D_{tt} \equiv \partial^2/\partial t^2$, and $D_{zt} \equiv \partial^2/\partial z\partial t$.

Consider the action of the most general linear second-order partial differential operator in z and t:

$$L = a_1 D_{tt} + a_2 D_{zt} + a_3 D_{zz} + a_4 D_z + a_5 D_t + a_6, \qquad a_i = a_i(z,t) \tag{A.3}$$

on the following time-dependent polynomial of degree N in z (see (2)),

$$\psi(z,t) = \prod_{n=1}^{N}[z - z_n(t)] = z^N + \sum_{m=1}^{N} c_m(t) z^{N-m}. \tag{A.4}$$

The restriction to second-order operators ensures that the *nonlinear* many-body problem for the zeros will have at most *two-body* interactions, which is the more physically interesting case. If many-body interactions are allowed, it is obvious that the class of such solvable systems that can be constructed with this method is larger. Since we want the evolution of the coefficients $c_m(t)$ to be a *linear autonomous* (hence *solvable*) system, we assume $a_i(z,t) = a_i(z)$ and without loss of generality we also set $a_1(z) = 1$.

We now examine the conditions on $a_i(z)$ such that the equation $L[\psi] = 0$ implies a linear system for $c_m(t)$. These conditions arise from imposing that L acting on $\psi(z,t)$ produces a polynomial of degree $N-1$. They are satisfied by a *second-order* differential operator in z and t only in the following cases:

- the *second-order* differential operator D_{tt};
- one of the following four differential operators of *first-order in t*:

$$D_t, \quad D_{zt}, \quad zD_{zt}, \quad z[zD_{zt} - (N-1)D_t]; \tag{A.5}$$

- a *second-order* differential operator in z that maps $\mathcal{P}_N$ to $\mathcal{P}_{N-1}$.

The characterization of vector spaces of linear (and nonlinear) differential operators in one and several variables of any given order that map $\mathcal{P}_N$ to $\mathcal{P}_{N-k}$ has been treated in [9]. Working out all the possible cases for a second-order differential operator in one variable that maps $\mathcal{P}_N$ to $\mathcal{P}_{N-1}$ is a simple exercise that produces the following result:

$$T_2^{+1} = z(zD_z - N + 1)(zD_z - N + 2), \tag{A.6a}$$

$$T_2^{0} = z^2 D_{zz} - N(N-1), \tag{A.6b}$$

$$T_2^{-1} = zD_{zz}, \tag{A.6c}$$

$$T_2^{-2} = D_{zz}, \tag{A.6d}$$

$$T_1^{0} = zD_z - N, \tag{A.6e}$$

$$T_1^{-1} = D_z. \tag{A.6f}$$

A linear combination of the eleven operators written above is precisely equation (1) of this paper, which was treated in section 2.3.3 of [2] (and several specific cases were then investigated in section 2.3.4). However, the question arises whether this is the most general class of PDEs in z and t which admit as solutions polynomials in z whose coefficients are functions of t that evolve according to a *linear* system. The recent discovery of the so-called *exceptional polynomial subspaces of* $\mathcal{P}_N$ shows that other PDEs with those properties exist (and therefore other *solvable* many-body problems) provided some constraints are imposed.

An exceptional polynomial subspace $\mathcal{M}_N^{(k)}$ of co-dimension k in $\mathcal{P}_N$ is defined by the property that some second-order differential operators that preserve $\mathcal{M}_N^{(k)}$ *do not preserve* $\mathcal{P}_N$. More specifically, consider a space $\mathcal{M}_N^{(k)} \subset \mathcal{P}_N$ generated by $N+1-k$ linearly independent polynomials, all of them of degree at most N:

$$\mathcal{M}_N^{(k)} = \mathrm{span}\{p_1(z), \ldots, p_{N+1-k}(z)\}. \tag{A.7}$$

We will say that $\mathcal{M}_N^{(k)}$ is an *exceptional polynomial subspace* of co-dimension k in $\mathcal{P}_N$ (for short, an X_k space) if $\mathcal{D}_2\big(\mathcal{M}_N^{(k)}\big) \not\subset \mathcal{D}_2(\mathcal{P}_N)$. Such exceptional subspaces do exist, and they provide novel differential operators with polynomial eigenfunctions. Exceptional subspaces have been first analysed in the context of quasi-exactly solvable potentials in quantum mechanics [7, 10]; they are also connected with the Darboux transformation [8] and with non-classical families of orthogonal polynomials [11]. X_1 spaces have been fully classified arriving at the result that there is essentially one such subspace up to projective transformations. This space is precisely

$$X_1 = \mathrm{span}\{1, z^2, z^3, \ldots, z^N\}, \tag{A.8a}$$

which can also be characterized as

$$X_1 = \{p \in \mathcal{P}_N | p'(0) = 0\}. \tag{A.8b}$$

It can be shown that $\dim \mathcal{D}_2(X_1) = 7$. The most interesting element of this space is the operator

$$D_{zz} - \frac{2}{z} D_z, \tag{A.9}$$

which preserves X_1 but not $\mathcal{P}_N$. The first class of models treated in this paper—see (6)—is related to this exceptional space X_1, and the novel term proportional to the constant A_4 in the many-body problem (6) is precisely that due to the inclusion of this operator. The constraint on the many-body problem (8) is precisely the one that defines the exceptional subspace X_1. The second class of many-body models treated above has a different origin, not related to the exceptional subspaces of $\mathcal{P}_N$, being instead associated with the role of the time derivative, see (A.5).

Exceptional subspaces of higher co-dimension exist but a full classification is not yet available. Some new many-body problems with constraints associated with X_2 spaces will be treated in a forthcoming publication [5], their main novelty being that the coefficients of the new differential operators are not only inverse powers but rational functions of z.

Appendix B

In this appendix, we justify the assertion made in section 5, that conditions (51*a*–51*d*) guarantee that the N-body model (6) with (10) is *entirely isochronous*.

Indeed, clearly this N-body problem is *entirely isochronous* if *all* the solutions of the corresponding linear problem (7) are *completely periodic* with the *same* period T (readers for whom this is not clear are advised to consult, say, [4]). Clearly a necessary and sufficient condition for this to happen is that the $2N$ eigenvalues $\lambda_m^{(\pm)}$, see (3), be *all integer multiples* of a common imaginary constant, say

$$\lambda_m^{(\pm)} = k_m^{(\pm)} \mathrm{i}\omega, \tag{B.1a}$$

with ω being a *positive* constant, $\omega = 2\pi/T > 0$, and the coefficients $k_m^{(\pm)}$ *all integers*, and moreover that

$$\lambda_m^{(+)} \neq \lambda_m^{(-)}, \tag{B.1b}$$

for all values of the index m, i.e. for $m = 1, \ldots, N$. Clearly the first of these two relations, (B.1*a*), can only be true if the quantity Δ_m^2, see (42*b*), is a perfect square for all values of m, so that

$$\Delta_m = \alpha + \beta m, \tag{B.2}$$

clearly entailing (see (42*b*))

$$\alpha = N A_5 + A_1, \qquad \beta^2 = A_5^2 - 4A_6, \tag{B.3a}$$

and (as we hereafter assume)

$$2A_3 + 2(2N-1)A_6 - A_5(N A_5 + A_1) = \alpha\beta. \tag{B.3b}$$

Via (3) these formulae yield

$$\lambda_m^{(+)} = \alpha + \frac{m}{2}(\beta - A_5), \qquad \lambda_m^{(-)} = -\frac{m}{2}(\beta + A_5). \tag{B.4}$$

Hence (B.1*a*) entails

$$\alpha = k_1 \mathrm{i}\omega, \qquad \beta - A_5 = 2j\mathrm{i}\omega, \qquad \beta + A_5 = 2k\mathrm{i}\omega, \tag{B.5}$$

with k_1, j, k being *integers*, and the (sum and difference of the) last two formulae entail

$$A_5 = k_2 \mathrm{i}\omega, \qquad \beta = k_3 \mathrm{i}\omega, \tag{B.6}$$

with k_2 and k_3 being *integers*. Hence the two equations (B.3*a*) yield

$$A_1 = (k_1 - Nk_2)\mathrm{i}\omega, \qquad A_6 = \frac{k_3^2 - k_2^2}{4}\omega^2, \tag{B.7}$$

and from (B.1*b*) one finally gets

$$A_3 = \tfrac{1}{2}(k_2 + k_3)\left[\left(N - \tfrac{1}{2}\right)(k_2 - k_3) - k_1\right]\omega^2. \tag{B.8}$$

The four formulae (51*e*) are thereby proven. And it is moreover clear that condition (B.1*b*) entails the requirement (51*e*), thereby completing the proof of the results reported in section 5.

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